Government College of Engineering, Keonjhar

Lecture Notes

Module-3

Vector Differential Calculus

vector and scalar functions and fields, Derivatives, Curves, tangents and arc Length, gradient, divergence, curl

Vector function: If a vector $\vec{F}$ is a function of a scalar variable t, then we write

$$\vec{F} = \vec{F}(t)$$

If the components of $F(t)$ along x-axis, y-axis and z-axis are $f_1(t), f_2(t), f_3(t)$ respectively then $\vec{F}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$.

Differentiation of vectors:

We define derivative of a vector function $\vec{F}(t)$ as $\frac{d\vec{F}}{dt} = \lim_{\delta t \to 0} \frac{\vec{F}(t+\delta t) - \vec{F}(t)}{\delta t}$

General rules of differentiation:

If ϕ is a scalar function and $\vec{F}, \vec{G}, \vec{H}$ are vector functions, then

(i) $\frac{d}{dt}(\vec{F} \pm \vec{G}) = \frac{d\vec{F}}{dt} \pm \frac{d\vec{G}}{dt}$

(ii) $\frac{d}{dt}(\vec{F}\phi) = \vec{F}\frac{d\phi}{dt} + \frac{d\vec{F}}{dt}\cdot\phi$

(iii) $\frac{d}{dt}(\vec{F}\cdot\vec{G}) = \vec{F}\cdot\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}\cdot\vec{G}$

(iv) $\frac{d}{dt}(\vec{F}\times\vec{G}) = \vec{F}\times\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}\times\vec{G}$.

(v) $\frac{d}{dt}[\vec{F}\vec{G}\vec{H}] = \left[\frac{d\vec{F}}{dt}\vec{G}\vec{H}\right] + \left[\vec{F}\frac{d\vec{G}}{dt}\vec{H}\right] + \left[\vec{F}\vec{G}\frac{d\vec{H}}{dt}\right]$

(vi) $\frac{d}{dt}\left[(\vec{F}\times\vec{G})\times\vec{H}\right] = \left(\frac{d\vec{F}}{dt}\times\vec{G}\right)\times\vec{H} + \left(\vec{F}\times\frac{d\vec{G}}{dt}\right)\times\vec{H} + (\vec{F}\times\vec{G})\times\frac{d\vec{H}}{dt}$

Ex: If $\vec{A} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, $\vec{B} = \sin t\,\hat{i} - \cos t\,\hat{j}$, find (i) $\frac{d}{dt}(\vec{A}\cdot\vec{B})$, (ii) $\frac{d}{dt}(\vec{A}\times\vec{B})$.

Solⁿ: (i) $\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \vec{A}\cdot\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\cdot\vec{B}$

$$= (5t^2\hat{i} + t\hat{j} - t^3\hat{k})\cdot\frac{d}{dt}(\sin t\,\hat{i} - \cos t\,\hat{j})$$

$$+ \frac{d}{dt}(5t^2\hat{i} + t\hat{j} - t^3\hat{k})\cdot(\sin t\,\hat{i} - \cos t\,\hat{j})$$

$$= (5t^2\hat{i} + t\hat{j} - t^3\hat{k})\cdot(\cos t\,\hat{i} + \sin t\,\hat{j})$$

$$+ (10t\hat{i} + \hat{j} - 3t^2\hat{k})\cdot(\sin t\,\hat{i} - \cos t\,\hat{j})$$

$$= (5t^2\cos t + t\sin t) + (10t\sin t - \cos t)$$

$$= 5t^2\cos t + 11t\sin t - \cos t.$$

(ii) $\frac{d}{dt}(\vec{A}\times\vec{B}) = \vec{A}\times\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\times\vec{B}$

$$= (5t^2\hat{i} + t\hat{j} - t^3\hat{k})\times\frac{d}{dt}(\sin t\,\hat{i} - \cos t\,\hat{j})$$

$$+ \frac{d}{dt}(5t^2\hat{i} + t\hat{j} - t^3\hat{k})\times(\sin t\,\hat{i} - \cos t\,\hat{j})$$

$$= (5t^2\hat{i} + t\hat{j} - t^3\hat{k})\times(\cos t\,\hat{i} + \sin t\,\hat{j})$$

$$+ (10t\hat{i} + \hat{j} - 3t^2\hat{k})\times(\sin t\,\hat{i} - \cos t\,\hat{j})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ \cos t & \sin t & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10t & 1 & -3t^2 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$$= \hat{i}\left(t^3\sin t\right) + \hat{j}\left(-t^3\cos t - \cancel{2t^5\sin t}\right) + \hat{k}\left(5t^2\sin t - t\cos t\right)$$

$$+ \hat{i}\left(-3t^2\cos t\right) + \hat{j}\left(-3t^2\sin t - 0\right) + \hat{k}\left(-10t\cos t - \sin t\right)$$

$$= \left(t^3\sin t - 3t^2\cos t\right)\hat{i} - t^2\left(t\cos t + 3\sin t\right)\hat{j} + \left[\left(5t^2 - 1\right)\sin t - 11t\cos t\right]\hat{k}$$

Scalar and vector point function:

If to each point $P(x,y,z)$ of a region R in space there corresponds a unique scalar $f(P)$, then f is called a scalar point function.

Ex: Temperature distribution in a heated body, density of a body and potential due to gravity are examples of scalar point function.

If to each point $P(x,y,z)$ of a region R in space there corresponds a unique vector $f(P)$, then f is called a vector point function.

Ex: The velocity of a moving fluid, gravitational force are the examples of a vector point function.

Vector Differential operator Del (∇):

The vector differential operator Del is denoted by ∇ and is defined by

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Gradient of a scalar function:

If $\phi(x,y,z)$ be a scalar function, then $\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$ is called the gradient of the scalar function ϕ and is denoted by grad ϕ or $\nabla.\phi$.

Thus, grad $\phi = \nabla.\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

Normal and Directional derivative:

(i) Normal: If $\phi(x,y,z) = c$ represents a family of surfaces for different values of the constant c, on differentiating ϕ, we get $d\phi = 0$

But $d\phi = \nabla.\phi.d\vec{r}$

So, $\nabla\phi.d\vec{r} = 0$.

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The scalar product of two vectors $\nabla\phi$ and $d\vec{r}$ being zero, $\nabla\phi$ and $d\vec{r}$ are perpendicular to each other. $d\vec{r}$ is the direction of tangent to the given surface.

thus $\nabla\phi$ is the vector normal to the surface $\phi(x,y,z) = c$.

(ii) <u>Directional Derivative</u>:

The component of $\nabla\phi$ in the direction of a vector $\vec{d}$ is equal to $\nabla\phi \cdot \hat{d}$ and is called the directional derivative of ϕ in the direction of $\vec{d}$.

<u>Ex</u>: If $\phi = 3x^2y - y^3z^2$, find grad ϕ at the point $(1,-2,-1)$.

<u>Soln</u>: grad $\phi = \nabla\phi$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \hat{i}(6xy) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z)$$

$\therefore$ grad ϕ at $(1,-2,-1) = \nabla\phi\big|_{(1,-2,-1)}$

$$= \hat{i}\{6(1)(-2)\} + \hat{j}\{3(1) - 3(4)(1)\} + \hat{k}\{-2\cdot(-8)\cdot(-1)\}$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}.$$

Ex: If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that

(i) $\text{grad}\, r = \dfrac{\vec{r}}{r}$ (ii) $\text{grad}\left(\dfrac{1}{r}\right) = -\dfrac{\vec{r}}{r^3}$

Soln: (i) $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow r = \sqrt{x^2+y^2+z^2}$

$\Rightarrow r^2 = x^2+y^2+z^2$

$\therefore 2r\dfrac{\partial r}{\partial x} = 2x \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\text{grad}\, r = \nabla r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}$$

$$= \hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\cdot\frac{z}{r}$$

$$= \frac{1}{r}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\vec{r}}{r}.$$

(ii) $\text{grad}\left(\dfrac{1}{r}\right) = \nabla\left(\dfrac{1}{r}\right)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \hat{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + \hat{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= -\frac{1}{r^3}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

$$= -\frac{\vec{r}}{r^3}.$$

Ex: Prove that $\nabla r^n = n r^{n-2}\vec{r}$, where $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

Sol^n:

$$r = \sqrt{x^2+y^2+z^2}$$

$$\therefore r^n = \left(x^2+y^2+z^2\right)^{n/2}$$

$$\frac{\partial r^n}{\partial x} = \frac{\partial}{\partial x}\left\{\left(x^2+y^2+z^2\right)^{n/2}\right\}$$

$$= \frac{n}{2}\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}\cdot 2x$$

$$= nx\left(x^2+y^2+z^2\right)^{\frac{n-2}{2}}$$

$$= nx\left(\sqrt{x^2+y^2+z^2}\right)^{n-2}$$

$$= nx\, r^{n-2}$$

Similarly, $\frac{\partial r^n}{\partial y} = ny\, r^{n-2}$

and $\frac{\partial r^n}{\partial z} = nz\, r^{n-2}$

$$\therefore \nabla r^n = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)r^n$$

$$= \hat{i}\frac{\partial}{\partial x}(r^n)+\hat{j}\frac{\partial}{\partial y}(r^n)+\hat{k}\frac{\partial}{\partial z}(r^n)$$

$$= \hat{i}\cdot\left(nx\,r^{n-2}\right)+\hat{j}\left(ny\,r^{n-2}\right)+\hat{k}\left(nz\,r^{n-2}\right)$$

$$= n r^{n-2}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

$$= n r^{n-2}\,\vec{r}$$

Ex: Find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ at $(1,-2,1)$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$.

Soln: Here, $\phi = x^2yz + 4xz^2$

$$\therefore \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2yz + 4xz^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2yz + 4xz^2) + \hat{j}\frac{\partial}{\partial y}(x^2yz + 4xz^2) + \hat{k}\frac{\partial}{\partial z}(x^2yz + 4xz^2)$$

$$= (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$$

Now, $$\nabla\phi\Big|_{(1,-2,1)} = \left\{2(1)(-2)(1) + 4(1)^2\right\}\hat{i} + (1\times1)\hat{j} + \left\{1(-2) + 8(1)(1)\right\}\hat{k}$$

$$= \hat{j} + 6\hat{k}$$

unit vector $$\hat{a} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{(2)^2 + (-1)^2 + (-2)^2}}$$

$$= \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$$

$\therefore$ The directional derivative of ϕ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$ is given by

$$\nabla\phi\Big|_{(1,-2,1)} \cdot \hat{a} = (\hat{j} + 6\hat{k}) \cdot \frac{(2\hat{i} - \hat{j} - 2\hat{k})}{3}$$

$$= \frac{1}{3}(-1 - 12) = -\frac{13}{3}.$$

Ex: Find the directional derivative of the function $\phi(x,y,z) = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where Q is the point $(5,0,4)$

Soln: Here, $\phi = x^2 - y^2 + 2z^2$

$$\therefore \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2-y^2+2z^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2-y^2+2z^2) + \hat{j}\frac{\partial}{\partial y}(x^2-y^2+2z^2) + \hat{k}\frac{\partial}{\partial z}(x^2-y^2+2z^2)$$

$$= 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$$

Now, $$\nabla\phi\Big|_{(1,2,3)} = \{2(1)\}\hat{i} - \{2(2)\}\hat{j} + \{4(3)\}\hat{k}$$

$$= 2\hat{i} - 4\hat{j} + 12\hat{k}$$

$$\vec{PQ} = \vec{Q} - \vec{P} = (5\hat{i} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= 4\hat{i} - 2\hat{j} + \hat{k}$$

unit vector $$\hat{a} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{(4)^2 + (-2)^2 + 1^2}}$$

$$= \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$$

$\therefore$ Directional derivative of ϕ along $\vec{PQ}$ is

$$\nabla\phi\Big|_{(1,2,3)} \cdot \hat{a} = (2\hat{i} - 4\hat{j} + 12\hat{k})\frac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{21}}$$

$$= \frac{8 + 8 + 12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

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Ex: Find the directional derivative of $\phi = 4e^{2x-y+z}$ at the point $(1,1,-1)$ in the direction towards the point $(-3,5,6)$.

Soln: Here, $\phi = 4e^{2x-y+z}$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(4e^{2x-y+z}\right)$$

$$= \hat{i}\frac{\partial}{\partial x}\left(4e^{2x-y+z}\right) + \hat{j}\frac{\partial}{\partial y}\left(4e^{2x-y+z}\right) + \hat{k}\cdot\frac{\partial}{\partial z}\left(4e^{2x-y+z}\right)$$

$$= 4\left[2e^{2x-y+z}\hat{i} - e^{2x-y+z}\hat{j} + e^{2x-y+z}\hat{k}\right]$$

$$= 4e^{2x-y+z}(2\hat{i} - \hat{j} + \hat{k})$$

$\therefore \nabla\phi$ at the point $(1,1,-1) = \nabla\phi\big|_{(1,1,-1)}$

$$= 4 \cdot e^{2-1-1} \cdot (2\hat{i} - \hat{j} + \hat{k})$$

$$= 4(2\hat{i} - \hat{j} + \hat{k})$$

Let $P(1,1,-1)$ and $Q(-3,5,6)$

Then $\vec{PQ} = (-3\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + \hat{j} - \hat{k})$

$$= -4\hat{i} + 4\hat{j} + 7\hat{k}$$

unit vector $\hat{a} = \dfrac{-4\hat{i} + 4\hat{j} + 7\hat{k}}{\sqrt{(-4)^2 + (4)^2 + (7)^2}}$

$$= \frac{-4\hat{i} + 4\hat{j} + 7\hat{k}}{\sqrt{16+16+49}}$$

$$= \frac{1}{9}(-4\hat{i} + 4\hat{j} + 7\hat{k})$$

$\therefore$ Directional derivative in the direction of $(-4\hat{i}+4\hat{j}+7\hat{k})$

$$= \nabla\phi\Big|_{(1,1,-1)} \cdot \hat{a}$$

$$= (8\hat{i}-4\hat{j}+4\hat{k}) \cdot \frac{(-4\hat{i}+4\hat{j}+7\hat{k})}{9}$$

$$= -\frac{20}{9}.$$

Angle between two surfaces

Let $\phi(x,y,z)=c_1$ and $\psi(x,y,z)=c_2$ are two surfaces.

Let θ be the angle between these two surfaces at any point (a,b,c)

Then $\cos\theta = \dfrac{n_1 \cdot n_2}{|n_1||n_2|}$

where $n_1 = \nabla f_1\Big|_{(a,b,c)}$

$n_2 = \nabla f_2\Big|_{(a,b,c)}$

and $f_1 = \phi(x,y,z) - c_1$

$f_2 = \psi(x,y,z) - c_2$

Ex: Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

Soln: Let $f_1 = x^2+y^2+z^2-9=0$

and $f_2 = x^2+y^2-z-3=0$.

$$\nabla f_1 = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-9)$$

$$= 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$\nabla f_1\Big|_{(2,-1,2)} = 4\hat{i}-2\hat{j}+4\hat{k}$$

$$\nabla f_2 = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2-z-3)$$

$$= 2x\hat{i}+2y\hat{j}-\hat{k}$$

$$\nabla f_2\Big|_{(2,-1,2)} = 4\hat{i}-2\hat{j}-\hat{k}$$

Let θ be the angle between the two surfaces at $(2,-1,2)$.

Then, $\cos\theta = \dfrac{n_1 \cdot n_2}{|n_1||n_2|}$

$$= \frac{(4\hat{i}-2\hat{j}+4\hat{k})(4\hat{i}-2\hat{j}-\hat{k})}{\sqrt{(4)^2+(-2)^2+(4)^2}\ \sqrt{(4)^2+(-2)^2+(-1)^2}}$$

$$= \frac{16+4-4}{\sqrt{36}\ \sqrt{21}} = \frac{16}{6\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

Divergence of a vector function:

The divergence of a vector point function $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ is denoted by $\mathrm{div}\, F$ or $\nabla \vec{F}$ and is defined as

$$\mathrm{div}\,\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

It is evident that $\mathrm{div}\,\vec{F}$ is a scalar function.
If $\mathrm{div}\,\vec{F} = 0$, $\vec{F}$ is called solenoidal.

Curl of a vector point function:

The curl of a vector point function $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ is denoted by curl $\vec{F}$ or $\nabla \times \vec{F}$ and is defined as

$$\mathrm{Curl}\, \vec{F} = \vec{\nabla} \times \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \hat{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

Curl $\vec{F}$ is a vector quantity.
If curl $\vec{F} = 0$, $\vec{F}$ is called irrotational vector field.

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Ex : Find the divergence and curl of the ~~function~~ vector $\vec{F} = xyz\hat{i} + 3x^2y\hat{j} + (xz^2 - y^2z)\hat{k}$ at $(2,-1,1)$.

Soln: Here, we have

$$\vec{F} = xyz\hat{i} + 3x^2y\hat{j} + (xz^2 - y^2z)\hat{k}$$

$$\text{div } \vec{F} = \nabla \vec{F}$$

$$= \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$= yz + 3x^2 + 2xz - y^2$$

$$\text{div}\vec{F} \text{ at } (2,-1,1) = -1 + 12 + 4 - 1 = 14$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 3x^2y & (xz^2 - y^2z) \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(xz^2 - y^2z) - \partial/\partial z(3x^2y)\right\}$$
$$+ \hat{j}\left\{\partial/\partial z(xyz) - \partial/\partial x(xz^2 - y^2z)\right\}$$
$$+ \hat{k}\left\{\partial/\partial x(3x^2y) - \partial/\partial y(xyz)\right\}$$

$$= \cancel{-2yz\hat{i} + (-z^2 + xy)\hat{j} + (6xy - xz)\hat{k}}$$

$$= -2yz\hat{i} + (xy - z^2)\hat{j} + (6xy - xz)\hat{k}$$

Curl $\vec{F}$ at $(2,-1,1)$

$$= \{-2(-1)(1)\}\hat{i} + \{2(-1)\cancel{(2)} - 1\}\hat{j} + \{6(2)(-1) - 2(1)\}\hat{k}$$

$$= 2\hat{i} - 3\hat{j} - 14\hat{k}.$$

Ex: If $\vec{v} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$, find the value of div $\vec{v}$.

Soln: We have, $\vec{v} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$

So div $\vec{v} = \nabla \vec{v}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \left\{\frac{(x^2+y^2+z^2)^{1/2}\cdot 1 - x\cdot\frac{2x}{\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)}\right\} + \left\{\frac{(x^2+y^2+z^2)^{1/2}\cdot 1 - y\cdot\frac{2y}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}\right\}$$

$$+ \left\{\frac{(x^2+y^2+z^2)^{1/2}\cdot 1 - z\cdot\frac{2z}{\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)}\right\}$$

$$= \frac{(x^2+y^2+z^2)-x^2}{(x^2+y^2+z^2)^{3/2}} + \frac{(x^2+y^2+z^2)-y^2}{(x^2+y^2+z^2)^{3/2}}$$

$$+ \frac{(x^2+y^2+z^2)-z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2}{\sqrt{x^2+y^2+z^2}}.$$

Ex: Show that $\operatorname{div}(\operatorname{grad} r^n) = n(n+1)r^{n-2}$, where $r = \sqrt{x^2+y^2+z^2}$

Solⁿ: We have, $r = \sqrt{x^2+y^2+z^2}$

$$r^2 = x^2+y^2+z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\operatorname{grad}(r^n) = \nabla r^n = n r^{n-2}\vec{r} \quad \text{[see previous ex.]}$$

$$= n r^{n-2}(x\hat{i}+y\hat{j}+z\hat{k})$$

Now, $\operatorname{div}(\operatorname{grad} r^n) = \operatorname{div}\left[n r^{n-2}(x\hat{i}+y\hat{j}+z\hat{k})\right]$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(n r^{n-2}x\hat{i} + n r^{n-2}y\hat{j} + n r^{n-2}z\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(n r^{n-2}.x) + \frac{\partial}{\partial y}(n r^{n-2}.y) + \frac{\partial}{\partial z}(n r^{n-2}z)$$

$$= \left\{ n r^{n-2} + n x (n-2) r^{n-3} \cdot \frac{\partial r}{\partial x} \right\} + \left\{ n r^{n-2} + n y (n-2) r^{n-3} \frac{\partial r}{\partial y} \right\}$$

$$+ \left\{ n r^{n-2} + n z (n-2) r^{n-3} \cdot \frac{\partial r}{\partial z} \right\}$$

$$= 3n r^{n-2} + n(n-2) \cdot r^{n-3} \left[x \cdot \frac{\partial r}{\partial x} + y \frac{\partial r}{\partial y} + z \cdot \frac{\partial r}{\partial z} \right]$$

$$= 3n r^{n-2} + n(n-2) r^{n-3} \left[x \left(\frac{x}{r}\right) + y \left(\frac{y}{r}\right) + z \left(\frac{z}{r}\right) \right]$$

$$= 3n r^{n-2} + n(n-2) r^{n-3} \cdot \left(\frac{x^2+y^2+z^2}{r} \right)$$

$$= 3n r^{n-2} + n(n-2) r^{n-3} \cdot \frac{r^2}{r} \quad \left[\because r^2 = x^2+y^2+z^2 \right]$$

$$= 3n \cdot r^{n-2} + n(n-2) r^{n-2}$$

$$= r^{n-2} (3n + n^2 - 2n)$$

$$= r^{n-2} (n^2 + n)$$

$$= n(n+1) r^{n-2}$$

If we put $n = -1$,

$$\therefore \text{div grad} \left(\frac{1}{r} \right) = -1(-1+1) r^{-1-2}$$

$$\therefore \nabla^2 \left(\frac{1}{r} \right) = 0.$$

Ex: If $\vec{V} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$, find the value of curl $\vec{V}$.

Soln: Curl $\vec{V} = \nabla \times \vec{V}$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}} \right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\right]$$

$$+ \hat{j}\left[\frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right]$$

$$= \hat{i}\left[\frac{-yz}{(x^2+y^2+z^2)^{3/2}} + \frac{yz}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$+ \hat{j}\left[\frac{zx}{(x^2+y^2+z^2)^{3/2}} - \frac{zx}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$+ \hat{k}\left[\frac{-xy}{(x^2+y^2+z^2)^{3/2}} + \frac{xy}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= 0.$$

Ex: Prove that $(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal and irrotational.

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Solⁿ: Let $\vec{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$

For Solenoidal, we have to prove $\nabla.\vec{F} = 0$.

$$\nabla\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\vec{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left\{(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}\right\}$$

$$= \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)$$

$$= -2 + 2x - 2x + 2$$

$$= 0$$

Thus $\vec{F}$ is solenoidal.

For irrotational, we have to prove curl $\vec{F} = 0$.

Now, curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2 - z^2 + 3yz - 2x) & (3xz + 2xy) & (3xy - 2xz + 2z) \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(3xz + 2xy)\right] + \hat{j}\left[\frac{\partial}{\partial z}(y^2 - z^2 + 3yz - 2x) - \frac{\partial}{\partial x}(3xy - 2xz + 2z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(3xz + 2xy) - \frac{\partial}{\partial y}(y^2 - z^2 + 3yz - 2x)\right]$$

$$= \hat{i}(3x-3x) + \hat{j}(-2z+3y-3y+2z) + \hat{k}(3z+2y-2y-3z)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= 0.$$

Thus, $\vec{F}$ is irrotational.

Ex: Determine the constants a and b such that the curl of vector $\vec{A} = (2xy+3yz)\hat{i} + (x^2+axz-4z^2)\hat{j} - (3xy+byz)\hat{k}$ is zero.

Soln:

$$\text{Curl } \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy+3yz) & (x^2+axz-4z^2) & -(3xy+byz) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-3xy-byz) - \frac{\partial}{\partial z}(x^2+axz-4z^2)\right] + \hat{j}\left[\frac{\partial}{\partial z}(2xy+3yz) - \frac{\partial}{\partial x}(-3xy-byz)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x^2+axz-4z^2) - \frac{\partial}{\partial y}(2xy+3yz)\right]$$

$$= \hat{i}(-3x-bz-ax+8z) + \hat{j}(3y+3y) + \hat{k}(2x+az-2x-3z)$$

$$= \{-x(3+a) + z(8-b)\}\hat{i} + 6y\hat{j} + z(a-3)\hat{k}$$

Curl $\vec{A} = 0$

$\Rightarrow \quad 3+a=0$ and $8-b=0, \qquad a-3=0$

$\Rightarrow \quad a=-3, \; b=8 \qquad \Rightarrow a=3$

$\therefore \; a=-3,3, \; b=8$

Ex: If a vector field is given by $\vec{F} = (x^2-y^2+x)\hat{i} - (2xy+y)\hat{j}$. Is this field irrotational? If so, find its scalar potential.

Soln: $\vec{F} = (x^2-y^2+x)\hat{i} - (2xy+y)\hat{j}$

Curl $\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2-y^2+x) & -(2xy+y) & 0 \end{vmatrix}$$

$$= \hat{i}\left[0 + \frac{\partial}{\partial z}(2xy+y)\right] + \hat{j}\left[\frac{\partial}{\partial z}(x^2-y^2+x) - 0\right] + \hat{k}\left[-\frac{\partial}{\partial x}(2xy+y) - \frac{\partial}{\partial y}(x^2-y^2+x)\right]$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= 0.$$

Hence $\vec{F}$ is irrotational.

To find the scalar potential function ϕ,

$$\vec{F} = \nabla \cdot \phi$$

$$d\phi = \frac{\partial \phi}{\partial x} \cdot dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} \cdot dz$$

$$= \left(\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} \right) \left(\hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz \right)$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \phi \cdot d\vec{r}$$

$$= \nabla \cdot \phi \cdot d\vec{r}$$

$$= \vec{F} \cdot d\vec{r}.$$

$$= \left[(x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j} \right] \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} \right)$$

$$= (x^2 - y^2 + x)\, dx - (2xy + y) \cdot dy.$$

$$\therefore \phi = \int \left[(x^2 - y^2 + x)\, dx - (2xy + y)\, dy \right] + c$$

$$= \int \left(x^2 \cdot dx + x\, dx - y\, dy - y^2\, dx - 2xy\, dy \right) + c$$

$$= \frac{x^3}{3} + \frac{x^2}{2} - \frac{y^2}{2} - xy^2 + c.$$

Hence, the scalar potential is

$$\frac{x^3}{3} + \frac{x^2}{2} - \frac{y^2}{2} - xy^2 + c.$$

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Ex: Find the scalar potential f for $\vec{A} = y^2\hat{i} + 2xy\hat{j} - z^2\hat{k}$

Soln: We have $\vec{A} = y^2\hat{i} + 2xy\hat{j} - z^2\hat{k}$

$$\text{Curl}\,\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy & -z^2 \end{vmatrix}$$

$$= \hat{i}(0) + \hat{j}(0) + \hat{k}(0)$$

$$= 0.$$

Hence, $\vec{A}$ is irrotational.

To find the scalar potential function f.

$$\vec{A} = \nabla f$$

$$df = \frac{\partial f}{\partial x}.dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= \nabla f . d\vec{r}$$

$$= (y^2\hat{i} + 2xy\hat{j} - z^2\hat{k})(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= y^2dx + 2xydy - z^2dz$$

$$\therefore f = \int y^2dx + 2xydy - z^2dz + c$$

$$= \int d(xy^2) - \int z^2dz + c$$

$$= xy^2 - \frac{z^3}{3} + c.$$

Del applied twice to point functions :

We have the following five formulae :

1). $\text{div. grad} f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

2). $\text{Curl grad} f = \nabla \times \nabla f = 0$.

3) $\text{div. Curl} \vec{F} = \nabla \cdot \nabla \times \vec{F} = 0$.

4) $\text{Curl Curl} \vec{F} = \text{grad. div.} \vec{F} - \nabla^2 \vec{F}$

i.e., $\nabla \times (\nabla \times \vec{F}) = \nabla \cdot (\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$

5). $\text{grad div} \vec{F} = \text{Curl Curl} \vec{F} + \nabla^2 \vec{F}$

i.e., $\nabla (\nabla \cdot \vec{F}) = \nabla \times (\nabla \times \vec{F}) + \nabla^2 \vec{F}$

Proofs : (1) $\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f$

$= \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$\nabla^2 f = \nabla \cdot (\nabla f) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$

$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$

$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

$= \nabla^2 f$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplacian operator.

(2) $\nabla \times \nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right] + \hat{j}\left[\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)\right] + \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right]$$

$$= \hat{i}\left[\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right] + \hat{j}\left[\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right] + \hat{k}\left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right]$$

$$= \hat{i}(0) + \hat{j}(0) + \hat{k}(0)$$

$$= 0$$

(3) $\text{div}(\text{curl } \vec{F}) = \nabla \cdot (\nabla \times \vec{F})$

Let $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$$\therefore \nabla(\nabla \times \vec{F}) = \nabla \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\hat{i}\,\partial/\partial x + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[\hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \hat{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\right]$$

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$$= \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right).$$

$$= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} + \frac{\partial^2 f_1}{\partial y \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial z \partial y}$$

$$= \left(\frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_3}{\partial y \partial x}\right) + \left(\frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_2}{\partial x \partial z}\right) + \left(\frac{\partial^2 f_1}{\partial y \partial z} - \frac{\partial^2 f_1}{\partial z \partial y}\right)$$

$$= 0.$$

(4). Let $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \hat{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$\therefore \text{Curl Curl } \vec{F} = \nabla \times (\nabla \times \vec{F})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) & \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) & \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\right]$$

$$+ \hat{j}\left[\frac{\partial}{\partial z}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\right]$$

$$= \hat{i}\left[\frac{\partial^2 f_2}{\partial y \partial x} - \frac{\partial^2 f_1}{\partial y^2} - \frac{\partial^2 f_1}{\partial z^2} + \frac{\partial^2 f_3}{\partial z \partial x}\right]$$

$$+ \hat{j}\left[\frac{\partial^2 f_3}{\partial z \partial y} - \frac{\partial^2 f_2}{\partial z^2} - \frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_1}{\partial x \partial y}\right]$$

$$+ \hat{k}\left[\frac{\partial^2 f_1}{\partial x \partial z} - \frac{\partial^2 f_3}{\partial x^2} - \frac{\partial^2 f_3}{\partial y^2} + \frac{\partial^2 f_2}{\partial y \partial z}\right]$$

$$= \hat{i}\left[\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial y \partial x} + \frac{\partial^2 f_3}{\partial z \partial x} - \left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\right)\right]$$

$$+ \hat{j}\left[\frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_1}{\partial x \partial y} + \frac{\partial^2 f_3}{\partial z \partial y} - \left(\frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_2}{\partial z^2}\right)\right]$$

$$+ \hat{k}\left[\frac{\partial^2 f_3}{\partial z^2} + \frac{\partial^2 f_1}{\partial x \partial z} + \frac{\partial^2 f_2}{\partial y \partial z} - \left(\frac{\partial^2 f_3}{\partial x^2} + \frac{\partial^2 f_3}{\partial y^2} + \frac{\partial^2 f_3}{\partial z^2}\right)\right]$$

$$= \hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f_1\right]$$

$$+ \hat{j}\left[\frac{\partial}{\partial y}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f_2\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f_3\right]$$

$$= \hat{i}\left[\frac{\partial}{\partial x}(\operatorname{div}\vec{F}) - \nabla^2 f_1\right]$$

$$+ \hat{j}\left[\frac{\partial}{\partial y}(\operatorname{div}\vec{F}) - \nabla^2 f_2\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial z}(\operatorname{div}\vec{F}) - \nabla^2 f_3\right]$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \operatorname{div}\vec{F}$$

$$- \nabla^2 \left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)$$

$$= \nabla.\left(\nabla \cdot \vec{F}\right) - \nabla^2\vec{F}$$

$$= \text{grad. div } \vec{F} - \nabla^2\vec{F}.$$

(5) $\overline{\text{Curl Curl } \vec{F} = \text{grad div}\vec{F} - \nabla^2\vec{F}}$

$\therefore$ grad div $\vec{F}$ = curl curl $\vec{F} + \nabla^2\vec{F}$

i.e. $\nabla(\nabla\cdot\vec{F}) = \nabla\times(\nabla\times\vec{F}) + \nabla^2\vec{F}$.

Government College of Engineering, Keonjhar

Lecture Notes

Module-4

Vector Integral Calculus

Line Integrals, Green Theorem, Surface integrals, Gauss theorem and Stokes Theorem (Without Proof)

Integration of vectors:

If two vectors $\vec{F}(t)$ and $\vec{G}(t)$ be such that

$$\frac{d\vec{G}(t)}{dt} = \vec{F}(t)$$

then $\vec{G}(t)$ is called an integral of $F(t)$ and we write

$$\int \vec{F}(t)\,dt = \vec{G}(t).$$

Its definite integral is $\int_a^b \vec{F}(t)\,dt = \vec{G}(b) - \vec{G}(a)$

Ex 1: Given $\vec{R}(t) = 3t^2\hat{i} + t\hat{j} - t^3\hat{k}$, evaluate

$$\int_0^1 \left(\vec{R} \times \frac{d^2\vec{R}}{dt^2}\right) dt.$$

Solⁿ: Here $\vec{R}(t) = 3t^2\hat{i} + t\hat{j} - t^3\hat{k}$

$$\therefore \frac{d\vec{R}(t)}{dt} = 6t\hat{i} + \hat{j} - 3t^2\hat{k}$$

$$\frac{d^2\vec{R}}{dt^2} = 6\hat{i} - 6t\hat{k}$$

$$\therefore \vec{R} \times \frac{d^2\vec{R}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3t^2 & t & -t^3 \\ 6 & 0 & -6t \end{vmatrix}$$

$$= \hat{i}(-6t^2 - 0) + \hat{j}(-6t^3 + 18t^3) + \hat{k}(0 - 6t)$$

$$= 6t^2\hat{i} + 12t^3\hat{j} - 6t\hat{k}$$

Now. $\int_0^1 \left(\vec{R} \times \frac{d^2\vec{R}}{dt^2}\right) dt = \int_0^1 \left(-6t^2\hat{i} + 12t^3\hat{j} - 6t\hat{k}\right) dt$

$$= \left[-\frac{6t^3}{3}\hat{i} + \frac{12t^4}{4}\hat{j} - \frac{6t^2}{2}\hat{k}\right]_0^1$$

$$= -2\hat{i} + 3\hat{j} - 3\hat{k}$$

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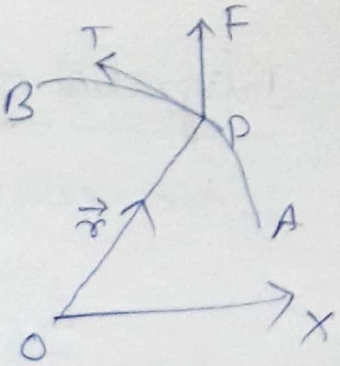
Line Integral :



Let $\vec{F}(x,y,z)$ be a vector function and AB be a given curve.

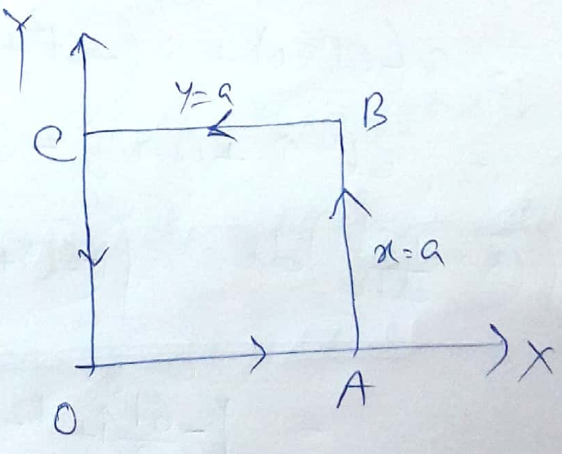
Line Integral of the vector function $\vec{F}$ along the curve AB is defined as integral of the component of $\vec{F}$ along the tangent to the curve AB.

$\therefore$ Line Integral $= \int_C \vec{F}\cdot d\vec{r}$.

Note : If $\vec{F}$ represents the variable force acting on a particle along arc AB, then the total work done $= \int_A^B \vec{F}\cdot d\vec{r}$.

Ex2: Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = x^2\hat{i} + xy\hat{j}$ and C is the boundary of the square in the plane $z=0$ and bounded by the lines $x=0$, $y=0$, $x=a$ and $y=a$.

Soln !



$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r}$$

Here, $\vec{r} = x\hat{i} + y\hat{j}$

$\therefore d\vec{r} = dx\hat{i} + dy\hat{j}$

and $\vec{F} = x^2\hat{i} + xy\hat{j}$

On OA, $y = 0$,

$$\therefore \vec{F}\cdot d\vec{r} = (x^2\hat{i} + 0\hat{j})(dx\hat{i} + 0\hat{j})$$
$$= x^2 dx$$

$$\therefore \int_{OA} \vec{F}\cdot d\vec{r} = \int_0^a x^2 dx = \int_0^a \frac{x^3}{3} = \frac{a^3}{3}. \quad \text{--- (1)}$$

On AB, $x = a$

$\therefore dx = 0$.

$$\therefore \vec{F}\cdot d\vec{r} = (a^2\hat{i} + ay\hat{j})(0\hat{i} + dy\hat{j})$$
$$= ay\,dy$$

$$\therefore \int_{AB} \vec{F}\cdot d\vec{r} = \int_0^a ay\,dy = a\left[\frac{y^2}{2}\right]_0^a$$
$$= \frac{a^3}{2}. \quad \text{--- (2)}$$

On BC. $y = a$, $\therefore dy = 0$

$$\therefore \vec{F}\cdot d\vec{r} = (x^2\hat{i} + xa\hat{j})(dx\hat{i} + 0\hat{j})$$
$$= x^2 dx$$

$$\therefore \int_{BC} \vec{F}\cdot d\vec{r} = \int_a^0 x^2\,dx = \left[\frac{x^3}{3}\right]_a^0 = -\frac{a^3}{3} \quad \text{---} (3)$$

On CO, $x=0$

$\therefore dx = 0$ and $\vec{F}\cdot d\vec{r} = 0\,dx + 0\,dy$
$= 0$ --- (4)

$$\therefore \int_{CO} \vec{F}\cdot d\vec{r} = 0 \quad \text{---} (4)$$

On adding (1), (2), (3) & (4), we get

$$\int_C \vec{F}\cdot d\vec{r} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0$$
$$= \frac{a^3}{2}$$

Ex 3: A vector field is given by $\vec{F} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the path $C: x=2t$, $y=t$, $z=t^3$ from $t=0$ to $t=1$.

Soln: Here $x=2t$, $y=t$, $z=t^3$

$\therefore dx = 2dt$, $dy = dt$, $dz = 3t^2 dt$

$$\vec{F}\cdot d\vec{r} = \{(2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}\}\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$
$$= (2y+3)\,dx + xz\,dy + (yz-x)\,dz$$
$$= \cancel{(2y+3)\cdot 2dt + xz\,dt}$$

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$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C (2y+3)\,dx + (xz)\,dy + (yz - x)\,dz$$

$$= \int_0^1 (2t+3)(2\,dt) + (2t)(t^3)\,dt + (t^4 - 2t)(3t^2\,dt)$$

$$= \int_0^1 (4t + 6 + 2t^4 + 3t^6 - 6t^3)\,dt$$

$$= \left[4\frac{t^2}{2} + 6t + 2\frac{t^5}{5} + 3\frac{t^7}{7} - 6\frac{t^4}{4}\right]_0^1$$

$$= \left[2t^2 + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{3}{2}t^4\right]_0^1$$

$$= 2 + 6 + \frac{2}{5} + \frac{3}{7} - \frac{3}{2}$$

$$= 7.328$$

Ex 4: If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$, where c is the curve in the xy-plane $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Sol^n: Since the particle moves in the xy plane $(z=0)$, we take, $\vec{r} = x\hat{i} + y\hat{j}$

$$\therefore d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C (3xy\hat{i} - y^2\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$

$$= \int_C (3xy\,dx - y^2\,dy)$$

Substituting, $y = 2x^2$, where x goes from 0 to 1, we get

$$\int_C \vec{F} \cdot d\vec{r} = \int_{x=0}^{1} 3x(2x^2) \cdot dx - (2x^2)^2 \cdot d(2x^2)$$

$$= \int_0^1 6x^3\,dx - 4x^4 \cdot 4x\,dx$$

$$= \int_0^1 (6x^3 - 16x^5)\,dx$$

$$= \left[6 \cdot \frac{x^4}{4} - 16\frac{x^6}{6}\right]_0^1$$

$$= -7/6.$$

Ex 5. If $\vec{A} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate the line integral $\oint \vec{A} \cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve C: $x = t, y = t^2, z = t^3$.

Soln: We have, $\int_C \vec{A} \cdot d\vec{r}$

$$= \int_C \left[(3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\right](dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

If $x = t$, $y = t^2$, $z = t^3$, then points $(0,0,0)$ to $(1,1,1)$ corresponds to $t = 0$ to $t = 1$.

Now, $\displaystyle\int_C \vec{A}\cdot d\vec{r} = \int_{t=0}^{t=1} \Big[(3t^2+6t^2)\,dt - 14t^2\cdot t^3\, d(t^2) + 20t(t^3)^2\cdot d(t^3)\Big]$

$$= \int_0^1 9t^2\,dt - 14t^5\cdot 2t\,dt + 20t\cdot t^6\cdot 3t^2\,dt$$

$$= \int_0^1 \left(9t^2 - 28t^6 + 60t^9\right) dt$$

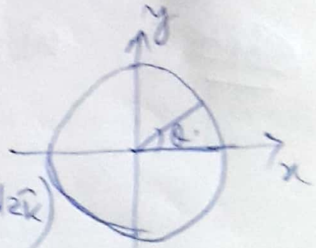
$$= \left[9\,\frac{t^3}{3} - 28\,\frac{t^7}{7} + 60\,\frac{t^{10}}{10}\right]_0^1$$

$$= \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1$$

$$= 3-4+6$$

$$= 5.$$

Ex 6 Compute $\displaystyle\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = \dfrac{\hat{i}y - \hat{j}x}{x^2+y^2}$ and C is the circle $x^2+y^2=1$ traversed counter clockwise.



$0 \le \theta \le 2\pi$

Soln: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C \left(\frac{y\hat{i} - x\hat{j}}{x^2+y^2}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$

$$= \int_C \frac{y\,dx - x\,dy}{x^2+y^2}$$

$$= \int_C (y\,dx - x\,dy) \qquad \left[\because x^2+y^2=1\right] \quad (1)$$

Parametric equations of the circle are

$$x = \cos\theta, \quad y = \sin\theta$$

Putting $x = \cos\theta$, $y = \sin\theta$

$dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$ in (1),

we get

$$\int_C \vec{F}\cdot d\vec{r} = \int_{\theta=0}^{\theta=2\pi} \sin\theta(-\sin\theta)\,d\theta - \cos\theta(\cos\theta\, d\theta)$$

$$= -\int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\,d\theta$$

$$= -\int_0^{2\pi} d\theta$$

$$= -2\pi.$$

Ex 7: If a force $\vec{F} = 2x^2y\,\hat{i} + 3xy\,\hat{j}$ displaces a particle in the xy-plane from $(0,0)$ to $(1,4)$ along a curve $y = 4x^2$. Find the work done.

Sol^n: Work done $= \int_C \vec{F}\cdot d\vec{r}$

$$= \int_C (2x^2y\,\hat{i} + 3xy\,\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_C (2x^2y\,dx + 3xy\,dy)$$

$$= \int_0^1 2x^2(4x^2)\cdot dx + 3x(4x^2)\cdot d(4x^2)$$

$$= \int_0^1 8x^4\,dx + 12x^3\cdot 8x\,dx$$

$$= \int_0^1 (8x^4 + 96x^4)\,dx = 104\left[\frac{x^5}{5}\right]_0^1$$

$$= \frac{104}{5}.$$

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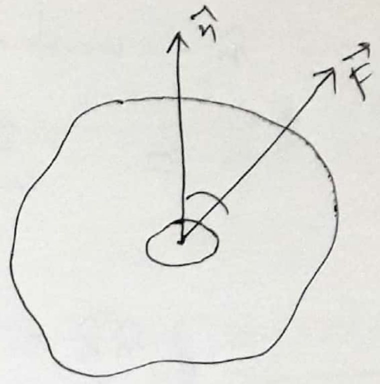
Surface Integral :



Let $\vec{F}$ be a vector function and S be a given surface.

Surface integral of a vector function $\vec{F}$ over the surface S is defined as the integral of the components of $\vec{F}$ along the normal to the surface.

Component of $\vec{F}$ along the normal $= \vec{F}\cdot\hat{n}$, where $\hat{n}$ is the unit normal vector to an element ds

and $\hat{n} = \dfrac{\text{grad} f}{|\text{grad} f|}$, $ds = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

Then, Surface Integral of $\vec{F}$ over the surface S.

$$= \iint_S \vec{F}\cdot\hat{n}\,ds$$

Ex 8: Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$, where $\vec{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $2x+y+2z = 6$ in the first octant.

Soln : A vector normal to the surface S is given by

$$\nabla(2x+y+2z-6) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x+y+2z-6)$$

$$= 2\hat{i} + \hat{j} + 2\hat{k}$$

$\hat{n}$ = unit vector normal to surface S

$$= \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{4+1+4}} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{3}$$

$$\therefore \hat{n}\cdot\hat{k} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{3}\cdot\hat{k} = \frac{2}{3}.$$

$$\therefore \iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

where R is the projection of S.

$$\text{Now, } \vec{A}\cdot\hat{n} = \left[(x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}\right]\cdot\left(\tfrac{2}{3}\hat{i}+\tfrac{1}{3}\hat{j}+\tfrac{2}{3}\hat{k}\right)$$

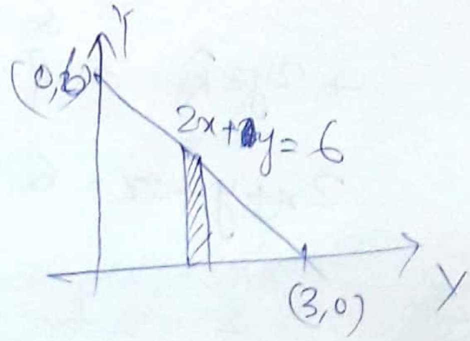
$$= \frac{2}{3}(x+y^2) - \frac{2}{3}x + \frac{4}{3}yz$$

$$= \frac{2}{3}y^2 + \frac{4}{3}yz$$

$$\therefore \iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \left(\frac{2}{3}y^2 + \frac{4}{3}yz\right)\cdot\frac{dx\,dy}{2/3} \quad \text{--- (1)}$$

$$\because 2x+y+2z=6$$

$$\Rightarrow z = \frac{6-2x-y}{2}$$



x varies from 0 to 3

y varies from 0 to (6-2x)

$2x+y=6$

$\Rightarrow \frac{x}{3}+\frac{y}{6}=1$

$\therefore$ from ①,

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \left\{\frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6-2x-y}{2}\right)\right\}\cancel{\tfrac{2}{3}}\cdot\frac{dx\,dy}{2/3}$$

$$= \int_0^3\int_0^{6-2x} \frac{2}{3}y\,(\cancel{y}+6-2x-\cancel{y})\cdot\cancel{\tfrac{2}{3}}\cdot\frac{dy\cdot dx}{2/3}$$

$$= \int_0^3\int_0^{(6-2x)} \frac{4}{3}y(3-x)\cdot\cancel{\tfrac{2}{3}}\,\frac{3}{2}\,dy\,dx$$

$$= 2\int_0^3 (3-x)\int_0^{6-2x} y\cdot dy\,dx$$

$$= 2\int_0^3 (3-x)\cdot\left[\frac{y^2}{2}\right]_0^{6-2x}\cdot dx$$

$$= \cancel{2}\int_0^3 (3-x)\cdot\frac{(6-2x)^2}{\cancel{2}}\,dx$$

$$= 4\int_0^3 (3-x)^2\,dx$$

$$= 4\left[\frac{(3-x)^4}{4(-1)}\right]_0^3$$

$$= -(0-81)$$

$$= 81.$$

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Ex 9: Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$, where $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is the part of the plane $2x+3y+6z=12$ included in the first octant.

Soln: Here $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$.

Given surface $f = 2x+3y+6z-12$

$\therefore$ Normal vector $= \nabla f$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x+3y+6z-12)$$

$$= 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$\therefore \hat{n}$ = unit normal at any point (x,y,z) of $2x+3y+6z=12$

$$= \frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{4+9+36}} = \frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})$$

$$\therefore ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})\cdot\hat{k}} = \frac{dx\,dy}{6/7}$$

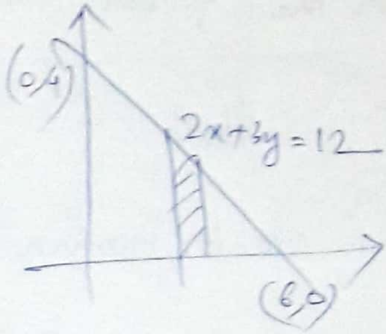
$$= \frac{7}{6}\,dx\,dy$$

Now, $$\iint \vec{A}\cdot\hat{n}\,ds = \iint (18z\hat{i} - 12\hat{j} + 3y\hat{k})\cdot\frac{(2\hat{i}+3\hat{j}+6\hat{k})}{7}\cdot\frac{7}{6}\,dx\,dy.$$

$$= \iint (6z - 6 + 3y)\,dx\,dy \quad \text{--- (1)}$$

$$2x+3y+6z=12$$

$$\Rightarrow z = \frac{12-2x-3y}{6}$$



$\therefore$ from ①, we get

x varies from 0 to 6

y varies from 0 to $\frac{12-2x}{3}$

$$\iint \vec{A}\cdot\hat{n}\,ds = \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (12-2x-3y-6+3y)\,dy.dx$$

$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} (6-2x)\,dy\,dx$$

$$= \int_0^6 (6-2x)\,dx \left[y \right]_0^{\frac{1}{3}(12-2x)}$$

$$= \int_0^6 (6-2x).\tfrac{1}{3}(12-2x).dx$$

$$= \int_0^6 \frac{4}{3}(3-x)(6-x)\,dx$$

$$= \frac{4}{3}\int_0^6 (18-9x+x^2)\,dx$$

$$= \tfrac{4}{3}\left[18x - \frac{9x^2}{2} + \frac{x^3}{3} \right]_0^6 = 24$$

Ex 10: Evaluate $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot \hat{n}\, ds$, where S is the surface of the sphere $x^2+y^2+z^2=a^2$ in the first octant.

Soln: Here $f = x^2+y^2+z^2-a^2$

Vector normal to the surface $= \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x^2+y^2+z^2-a^2)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}.$$

$$\therefore \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{2\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a} \quad [\because x^2+y^2+z^2=a^2]$$

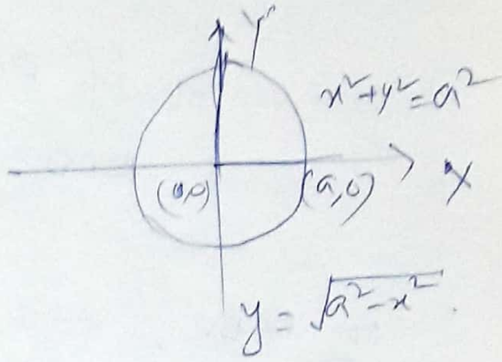
$$\hat{n}\cdot\hat{k} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}\cdot\hat{k} = \frac{z}{a}.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, ds = \iint_S \vec{F}\cdot\hat{n}\, \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \iint (yz\hat{i}+zx\hat{j}+xy\hat{k})\cdot\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{a}\cdot\frac{dx\,dy}{\frac{z}{a}}.$$

$$= \iint \frac{3xyz}{a} \cdot \frac{a}{z} \cdot dx\,dy .$$

$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} 3xy\,dy\,dx$$



$$= 3\int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} \cdot dx$$

$$= \frac{3}{2}\int_0^a x\,(a^2-x^2)\cdot dx$$

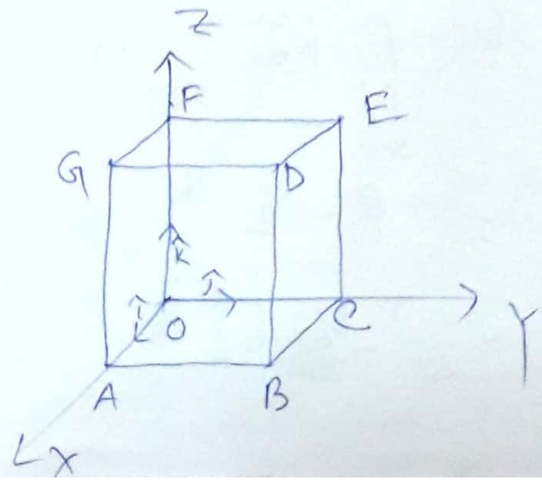
$$= \frac{3}{2}\left(a^2\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^a$$

$$= \frac{3}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right)$$

$$= \frac{3a^4}{8} .$$

Ex 11: Show that $\iint_S \vec{F}\cdot\hat{n}\,ds = 3/2$, where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded by the planes, $x=0, x=1, y=0, y=1, z=0, z=1$.

Soln:



Surface	outward normal	ds
OABC	$-\hat{k}$	$dx\,dy$, $z=0$
DEFG	$\hat{k}$	$dx\,dy$, $z=1$
OAGF	$-\hat{j}$	$dx\,dz$, $y=0$
BCED	$\hat{j}$	$dx\,dz$, $y=1$
ABDG	$\hat{i}$	$dy\,dz$, $x=1$
OCEF	$-\hat{i}$	$dy\,dz$, $x=0$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{OABC} \vec{F}\cdot\hat{n}\,ds + \iint_{DEFG} \vec{F}\cdot\hat{n}\,ds + \iint_{OAGF} \vec{F}\cdot\hat{n}\,ds$$

$$+ \iint_{BCED} \vec{F}\cdot\hat{n}\,ds + \iint_{ABDG} \vec{F}\cdot\hat{n}\,ds + \iint_{OCEF} \vec{F}\cdot\hat{n}\,ds \quad \text{——— (1)}$$

<u>on OABC</u>, $z=0$.

$$\therefore \iint_{OABC} \vec{F}\cdot\hat{n}\,ds = \iint_{OABC} \left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)\cdot(-\hat{k})\,dxdy$$

$$= \int_0^1\int_0^1 (-yz)\,dxdy$$

$$= 0 \quad [\because z=0].$$

<u>on DEFG, $z=1$</u>.

$$\therefore \iint_{DEFG} \vec{F}\cdot\hat{n}\,ds = \iint_{DEFG} \left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)\cdot\hat{k}\,dxdy$$

$$= \int_0^1\int_0^1 yz\,dxdy$$

$$= \int_0^1\int_0^1 y\,dxdy \quad [\because z=1]$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 \frac{1}{2}\cdot dx$$

$$= \frac{1}{2}\left[x\right]_0^1 = \frac{1}{2}.$$

$$\iint_{OAGF} \vec{F}\cdot\hat{n}\,ds = \iint_{OAGF} \left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)(-\hat{j})\cdot dxdz$$

$$= \iint_{OAGF} y^2\,dxdz$$

$$= \int_0^1\int_0^1 y^2\,dxdz$$

$$= 0 \quad [\because y=0].$$

$$\iint_{BCED} \vec{F}\cdot\hat{n}\,ds = \iint_{BCED} \left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)(\hat{j})\,dxdz$$

$$= \iint_{BCED} -y^2\,dxdz$$

$$= \int_0^1\int_0^1 -y^2\,dxdz$$

$$= -\int_0^1 dx \cdot \int_0^1 dz \quad [\because y=1]$$

$$= -[x]_0^1 \cdot [z]_0^1$$

$$= -1.$$

$$\iint_{ABDG} \vec{F}\cdot\hat{n}\,ds = \iint_{ABDG} \left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)\hat{i}\,dydz$$

$$= \int_0^1\int_0^1 4xz\,dydz$$

$$= \int_0^1\int_0^1 4z\,dydz \quad [\because x=1]$$

$$= 4\left[y\right]_0^1 \cdot \left[\frac{z^2}{2}\right]_0^1$$

$$= 4 \cdot 1 \cdot \tfrac{1}{2}$$

$$= 2.$$

$$\iint_{OCEF} \vec{F}\cdot\hat{n}\,ds = \iint_{OCEF} \left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)\cdot(-\hat{i})\,dy\,dz$$

$$= \int_0^1\int_0^1 -4xz\,dy\,dz$$

$$= 0 \quad [\because x = 0]$$

$\therefore$ from ①, we get

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 0 + \tfrac{1}{2} + 0 - 1 + 2 + 0$$

$$= 3/2.$$

Volume Integral.

Let $\vec{F}$ be a vector point function and volume v enclosed by a closed surface.

the volume integral $= \iiint_V \vec{F}\, dv.$

Ex 12: If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, evaluate $\iiint_V \vec{F}\, dv$, where v is the region bounded by the surfaces $x=0,\ y=0,\ x=2,\ y=4,\ z=x^2,\ z=2$.

Sol^n : $\iiint_V \vec{F}\cdot dv = \iiint (2z\hat{i} - x\hat{j} + y\hat{k})\,dx\,dy\,dz$

$$= \int_0^2\int_0^4\int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})\,dz\,dy\,dx$$

$$= \int_0^2\int_0^4 \left[2\frac{z^2}{2}\hat{i} - xz\hat{j} + yz\hat{k}\right]_{x^2}^2 \cdot dy \cdot dx$$

$$= \int_0^2\int_0^4 \left[z^2\hat{i} - xz\hat{j} + yz\hat{k}\right]_{x^2}^2 dy\,dx$$

$$= \int_0^2\int_0^4 \left[(4-x^4)\hat{i} - (2x-x^3)\hat{j} + (2y - x^2y)\hat{k}\right] dy\,dx$$

$$= \int_0^2 \left[(4-x^4)y\hat{i} - (2x-x^3)y\hat{j} + \left(2\frac{y^2}{2} - x^2\frac{y^2}{2}\right)\hat{k}\right]_0^4 \cdot dx$$

$$= \int_0^2 \left\{4(4-x^4)\hat{i} - 4(2x-x^3)\hat{j} + \left(16 - \frac{16x^2}{2}\right)\hat{k}\right\} dx$$

$$= \int_0^2 \left\{(16-4x^4)\hat{i} - (8x-4x^3)\hat{j} + (16-8x^2)\hat{k}\right\} dx$$

$$= \left[\left(16x - \frac{4x^5}{5}\right)\hat{i} - \left(8\frac{x^2}{2} - 4\frac{x^4}{4}\right)\hat{j} + \left(16x - \frac{8x^3}{3}\right)\hat{k}\right]_0^2$$

$$= \left[\left(16x - \frac{4x^5}{5}\right)\hat{i} - (4x^2 - x^4)\hat{j} + \left(16x - \frac{8}{3}x^3\right)\hat{k}\right]_0^2$$

$$= \left(32 - \frac{128}{5}\right)\hat{i} - (16-16)\hat{j} + \left(32 - \frac{64}{3}\right)\hat{k}$$

$$= \frac{32}{5}\hat{i} + \frac{32}{3}\hat{k} = \frac{32}{15}(3\hat{i} + 5\hat{k})$$

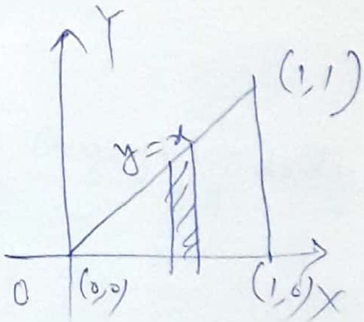
Green's theorem

Statement: If $\phi(x,y)$, $\psi(x,y)$, $\frac{\partial \phi}{\partial y}$ and $\frac{\partial \psi}{\partial x}$ be continuous functions over a region R bounded by simple closed curve C in xy plane, then

$$\oint_C (\phi\,dx + \psi\,dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy.$$

Ex 13: Using Green's theorem, evaluate $\oint_C (x^2 y\,dx + x^2 dy)$, where C is the boundary described counter clockwise of the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$.

Soln: By Greens theorem, we have



$$\oint_C (\phi\,dx + \psi\,dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$$

$$\Rightarrow \oint_C (x^2 y\,dx + x^2 dy) = \iint_R \left\{\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(x^2 y)\right\} dx\,dy$$

$$\Rightarrow \int_0^1 (x^2 y\,dx + x^2 dy) = \int_{x=0}^{1}\int_{y=0}^{x} (2x - x^2)\,dy\,dx$$

$$= \int_0^1 (2x - x^2)\cdot[y]_0^x \cdot dx$$

$$= \int_0^1 (2x - x^2)\cdot x \cdot dx$$

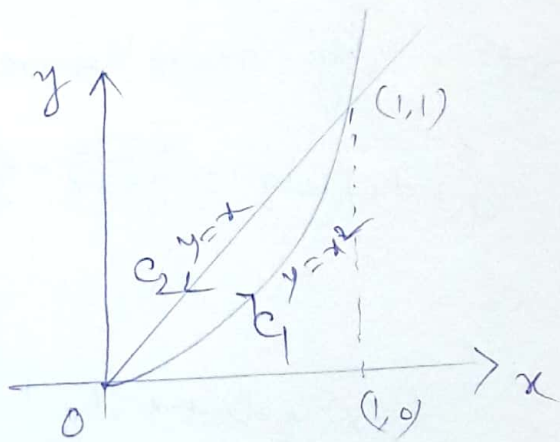
$$= \int_0^1 (2x^2 - x^3)\, dx$$

$$= \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{5}{12}$$

Ex 14: Verify Green's theorem for $\oint_C (xy+y^2)dx + x^2 dy$, where C is bounded by $y=x$ and $y=x^2$.

Solⁿ: Here, $\varphi = xy+y^2$, $\psi = x^2$

$$\therefore \int_C (\varphi\, dx + \psi\, dy) = \int_{C_1} (\varphi\, dx + \psi\, dy) + \int_{C_2} (\varphi\, dx + \psi\, dy)$$



Along C_1, $y=x^2$ and x varies from 0 to 1.

$$\therefore \int_{C_1} (xy+y^2)dx + x^2 dy = \int_0^1 \left[\left\{ x(x^2) + (x^2)^2 \right\} dx + x^2\, d(x^2) \right]$$

$$= \int_0^1 (x^3 + x^4 + 2x^3)\, dx$$

$$= \int_0^1 (3x^3 + x^4)\, dx = \frac{19}{20}.$$

Along C_2, $y = x$ and x varies from 1 to 0

$$\therefore \oint_{C_2} (xy+y^2)\,dx + x^2\,dy = \int_1^0 \left\{ (x \cdot x + x^2)\,dx + x^2\,d(x) \right\}$$

$$= \int_1^0 3x^2\,dx = -1.$$

Thus, $\oint_C (\phi\,dx + \psi\,dy) = \oint_C (xy+y^2)\,dx + x^2\,dy$

$$= \frac{19}{20} - 1 = -\frac{1}{20} \quad \text{——— (1)}$$

Now, by Greens theorem

$$\oint_C (\phi\,dx + \psi\,dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\,dy$$

$$= \iint_R \left\{ \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy+y^2) \right\} dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (2x - x - 2y)\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} (x - 2y)\,dy\,dx$$

$$= \int_0^1 \left[xy - y^2 \right]_{x^2}^{x} dx$$

$$= \int_0^1 \left(x^2 - x^2 - x^3 + x^4 \right) dx$$

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$$= \int_0^1 (x^4 - x^3)\, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = -\frac{1}{20} \quad \text{——— (2)}$$

Hence, Green's theorem is verified from the equality of (1) and (2).

Ex 15: Verify Green's theorem in the plane for $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$, where C is the boundary of the region bounded by $x \geq 0$, $y \leq 0$ and $2x - 3y = 6$

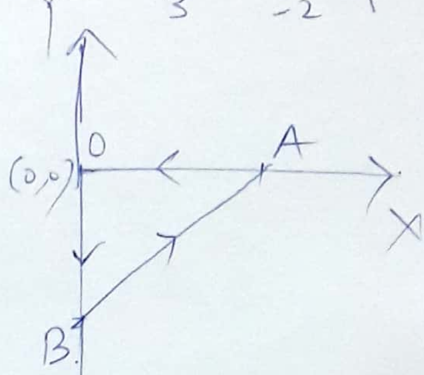
Soln:

$$\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$2x - 3y = 6$

$\frac{x}{3} + \frac{y}{-2} = 1$

$$= \int_{OB} + \int_{BA} + \int_{AO}.$$



Along OB, $x = 0$ and y varies from 0 to -2

$$\therefore \int_{OB} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_0^{-2} \{3(0)^2 - 8y^2\}\,d(0) + \{4y - 6(0)y\}\,dy$$

$$= \int_0^{-2} 4y\,dy = \left[2y^2\right]_0^{-2} = 8.$$

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Along BA, $x = \frac{6+3y}{2}$ and y varies from -2 to 0

$\therefore dx = \frac{3}{2} dy.$

$$\therefore \int_{BA} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= \int_{-2}^{0} \left\{3\left(\frac{6+3y}{2}\right)^2 - 8y^2\right\}\frac{3}{2}.dy + \left\{4y - 6\left(\frac{6+3y}{2}\right)y\right\}dy$$

$$= \int_{-2}^{0} \left[\frac{9}{8}(6+3y)^2 - 12y^2 + 4y - 18y - 9y^2\right] dy$$

$$= \int_{-2}^{0} \left[\frac{9}{8}(6+3y)^2 - 21y^2 - 14y\right] dy$$

$$= \left[\frac{9}{8}\frac{(6+3y)^3}{3\times 3} - 21\frac{y^3}{3} - 14\frac{y^2}{2}\right]_{-2}^{0}$$

$$= \left[\frac{1}{8}(6+3y)^3 - 7y^3 - 7y^2\right]_{-2}^{0}$$

$$= \frac{6^3}{8} - 56 + 28$$

$$= 27 - 56 + 28$$

$$= -1.$$

Along AO, $y = 0$ and x varies from 3 to 0.

$$\therefore \int_{AO} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= \int_{3}^{0} (3x^2 - 0)dx + (0)d(0)$$

$$= \left[x^3\right]_3^0 = -27.$$

$\therefore \oint (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

$= 8 - 1 - 27$

$= -20$. ——— (1)

Now, by Green's theorem, we have

$$\oint (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2)\right] dx\,dy$$

$$= \iint_R (-6y + 16y)\,dx\,dy$$

$$= \iint_R 10y\,dx\,dy$$

$$= 10 \int_0^3 \int_{\frac{2x-6}{3}}^{0} y\,dy\,dx$$

$$= 10 \int_0^3 dx \left[\frac{y^2}{2}\right]_{\frac{2x-6}{3}}^{0}$$

$$= 10 \cdot \int_0^3 \left(-\frac{1}{2}\right) \frac{(2x-6)^2}{9}\,dx$$

$$= -\frac{5}{9} \cdot \frac{(2x-6)^3}{3\times 2}\Bigg|_0^3$$

$$= -\frac{5}{54}(0 + 6^3)$$

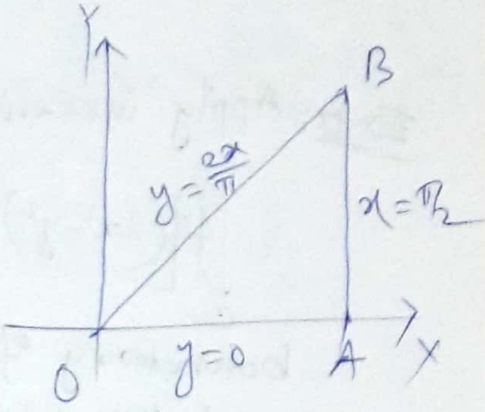
$= -20$. ——— (2)

$\therefore$ Green's theorem verified.

Ex 16: Using Green's theorem, evaluate $\oint_C (y-\sin x)\,dx + \cos x\,dy$, where c is the plane triangle enclosed by the lines $y=0$, $x=\pi/2$ and $y=\frac{2}{\pi}x$.

Sol^n: Here, $\phi = y-\sin x$

$\psi = \cos x$



By Green's theorem,

$$\oint_C (y-\sin x)\,dx + \cos y\,dy$$

$$= \iint_R \left\{ \frac{\partial}{\partial x}(\cos x) - \frac{\partial}{\partial y}(y-\sin x) \right\} dx\,dy$$

$$= \int_{x=0}^{\pi/2} \int_{y=0}^{\frac{2x}{\pi}} (-\sin x - 1)\,dy\,dx$$

$$= -\int_0^{\pi/2} (\sin x+1)\cdot\left[y\right]_0^{\frac{2x}{\pi}}\,dx$$

$$= -\int_0^{\pi/2} (\sin x+1)\cdot\frac{2x}{\pi}\,dx$$

$$= -\frac{2}{\pi}\int_0^{\pi/2} x(\sin x+1)\,dx$$

$$= -\frac{2}{\pi}\left\{ x(-\cos x+x)\Big|_0^{\pi/2} - \int_0^{\pi/2} 1\cdot(-\cos x+x)\,dx \right\}$$

[Integrating by parts]

$$= -\frac{2}{\pi}\left\{ \frac{\pi^2}{4} - \left[-\sin x + \frac{x^2}{2} \right]_0^{\pi/2} \right\}$$

$$= -\pi/2 + \frac{2}{\pi}\left(-1 + \frac{\pi^2}{8}\right)$$

$$= -\left(\pi/4 + \frac{2}{\pi}\right).$$

Ex 17: Apply Green's theorem to evaluate

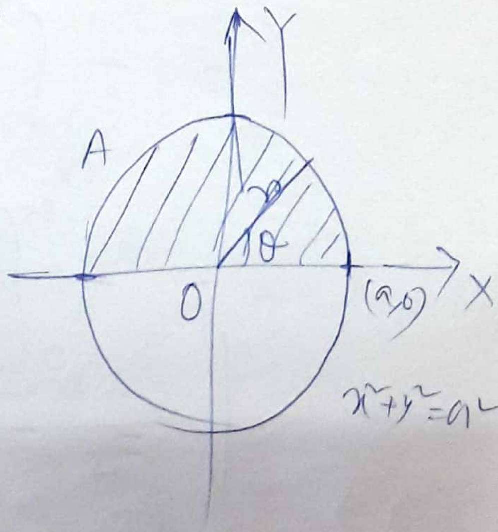
$$\oint_C \left[(2x^2-y^2)\,dx + (x^2+y^2)\,dy\right],$$ where c is the boundary of the area enclosed by the x-axis and the upper half of the circle $x^2+y^2=a^2$.

Solⁿ: By Green's theorem,

$$\oint_C \left[(2x^2-y^2)\,dx + (x^2+y^2)\,dy\right]$$

$$= \iint_A \left[\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(2x^2-y^2)\right] dx\,dy$$

$$= \iint_A 2(x+y)\,dx\,dy$$



Let $x = r\cos\theta$, $y = r\sin\theta$

Then, r varies from 0 to a

θ varies from 0 to π

and $dx\,dy = r\,dr\,d\theta$

$$\therefore \oint_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$$

$$= 2\int_0^a \int_0^\pi r(\cos\theta + \sin\theta)\cdot r\,dr\,d\theta$$

$$= 2\int_0^a r^2\,dr \cdot \int_0^\pi (\cos\theta+\sin\theta)\,d\theta = \frac{4a^3}{3}.$$

Stoke's theorem (Relation between line and surface integrals)

If S be an open surface bounded by a closed curve C and $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be any continuously differentiable vector point function, then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,ds$$

where, $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$ is a unit external normal to any surface ds.

Ex 18: Using Stoke's theorem, evaluate

$$\oint_C \left[(2x-y)\,dx - yz^2\,dy - y^2z\,dz\right],$$ where C is the circle $x^2+y^2=1$, corresponding to the surface of sphere of unit radius.

Solⁿ: $\oint_C (2x-y)\,dx - yz^2\,dy - y^2z\,dz$

$$= \oint_C \left[(2x-y)\hat{i} - yz^2\hat{j} - y^2z\,\hat{k}\right]\cdot\left[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right]$$

By Stoke's theorem,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,ds \quad \text{--- (1)}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \hat{i}\left(-2yz+2yz\right) - \hat{j}\left(0-0\right) + \hat{k}\left(0+1\right)$$

$$= \hat{k}$$

Here $ds = \frac{dx\,dy}{|\hat{n}\hat{k}|}$.

Putting the value of curl $\vec{F}$ in ①, we get

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{F}\cdot\hat{n}\, ds$$

$$= \iint_S \hat{k}\cdot\hat{n} \cdot \frac{dx\,dy}{\hat{n}\hat{k}}$$

$$= \iint_S dx\,dy$$

= Area of the circle of unit radius

$$= \pi(1)^2$$

$$= \pi.$$

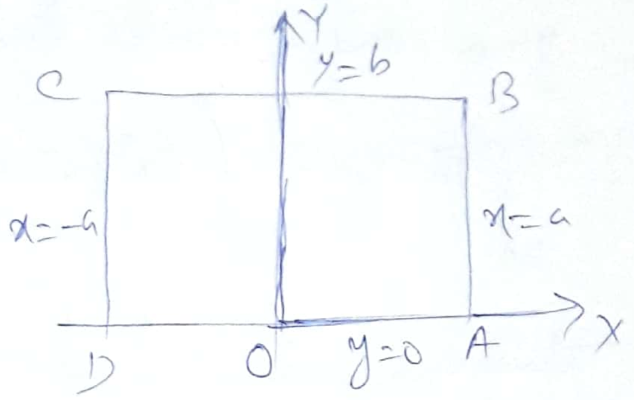
Ex 19: Verify Stoke's theorem for $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by the lines $x = \pm a$, $y=0$, $y=b$.

Soln: Let ABCD be the given rectangle

$$\oint_{ABCD} \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r} + \int_{DA} \vec{F}\cdot d\vec{r}$$

and $\vec{F}\cdot d\vec{r} = \left[(x^2+y^2)\hat{i} - 2xy\hat{j}\right]\cdot(dx\hat{i}+dy\hat{j})$

$$= (x^2+y^2)\,dx - 2xy\,dy$$



Along AB, $x=a$ (i.e, $dx=0$) and y varies from 0 to b.

$$\therefore \int_{AB} \vec{F}\cdot d\vec{r} = \int_0^b (a^2+y^2)\,d(0) - 2ay\,dy$$

$$= \int_0^b -2ay\,dy$$

$$= -2a\cdot\left[\frac{y^2}{2}\right]_0^b$$

$$= -ab^2$$

Along BC, $y=b$ (i.e, $dy=0$) and x varies from ~~a to (-a)~~ a to (-a)

$$\therefore \int_{BC} \vec{F}\cdot d\vec{r} = \int_a^{-a} (x^2+b^2)\,dx$$

$$= \left[\frac{x^3}{3} + b^2x\right]_a^{-a}$$

$$= -\frac{2a^3}{3} - 2ab^2$$

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Along CD, $x = -a$ and y varies from b to 0.

$$\therefore \int_{CD} \vec{F}\cdot d\vec{r} = \int_b^0 2ay\, dy$$

$$= 2a\left[\frac{y^2}{2}\right]_b^0$$

$$= -ab^2$$

Along DA, $y = 0$ and x varies from $-a$ to a

$$\therefore \int_{DA} \vec{F}\cdot d\vec{r} = \int_{-a}^{a} (x^2+0^2)\, dx$$

$$= \left[\frac{x^3}{3}\right]_{-a}^{a}$$

$$= \frac{2a^3}{3}.$$

Thus $$\int_{ABCD} \vec{F}\cdot d\vec{r} = \left(-ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3}\right)$$

$$= -4ab^2 \quad \text{———} \quad (1)$$

Now $$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(-2y-2y)$$

$$= -4y\hat{k}$$

$$\therefore \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\, ds$$

$$= \iint_S (-4y\hat{k})\cdot\hat{k}\, dx\, dy$$

$$= \int_{y=0}^{b}\int_{x=-a}^{a} -4y.\, dx\, dy$$

$$= -4 \int_{x=-a}^{a} \int_{y=0}^{b} y \, dy \, . \, dx$$

$$= -4 \int_{-a}^{a} \left[\frac{y^2}{2}\right]_0^b . \, dx$$

$$= -4 . \int_{-a}^{a} \frac{b^2}{2} . \, dx$$

$$= -2b^2 \int_{-a}^{a} . \, dx$$

$$= -2b^2 . [x]_{-a}^{a}$$

$$= -4ab^2 .$$

Ex 20: Evaluate $\int_C \vec{F} . d\vec{r}$, where $\vec{F} = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$, and C is the curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$.

Soln: $\oint_C \vec{F} . d\vec{r} = \iint_S \text{curl} \vec{F} . \hat{n} \, ds$

$$= \iint_S (\nabla \times \vec{F}) . \hat{n} \, ds.$$

$$\text{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(1+2y)$$

$$= (1+2y)\hat{k} .$$

The given plane is $y+z-2=0$

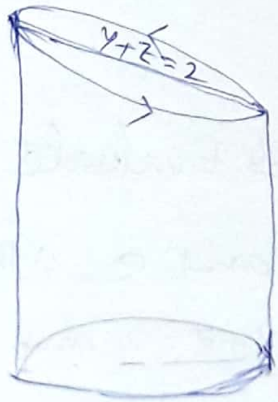
Normal vector $\phi = \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(y+z-2)$$

$$= \hat{i}(0)+\hat{j}(1)+\hat{k}(1)$$

$$= \hat{j}+\hat{k}$$

$\therefore$ unit normal vector $\hat{n} = \dfrac{\hat{j}+\hat{k}}{\sqrt{1+1}} = \dfrac{\hat{j}+\hat{k}}{\sqrt{2}}$



$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$\hat{n}\cdot\hat{k} = \frac{\hat{j}+\hat{k}}{\sqrt{2}}\cdot\hat{k} = \frac{1}{\sqrt{2}}$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \iint_S (1+2y)\,\hat{k}\cdot\frac{(\hat{j}+\hat{k})}{\sqrt{2}}\cdot\frac{dx\,dy}{\frac{1}{\sqrt{2}}}$$

$$= \iint_S (1+2y)\,\frac{1}{\sqrt{2}}\cdot\sqrt{2}\cdot dx\,dy$$

$$= \iint_S (1+2y)\,dx\,dy$$

Let $x = r\cos\theta,\ y = r\sin\theta$

Then $dx\,dy = r\,dr\,d\theta$

r varies from 0 to 1

θ varies from 0 to 2π.

Gauss Divergence Theorem

(Relation between surface and volume integrals)

Statement: The surface integral of the normal component of a vector function $\vec{F}$ taken around a closed surface S is equal to the integral of the divergence of $\vec{F}$ taken over the volume V enclosed by the surface S.

Mathematically, $$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{F}\, dv$$

Ex. 22: Use Gauss divergence theorem to show that

$$\iint_S \nabla(x^2+y^2+z^2)\, d\vec{S} = 6V,$$ where S is any closed surface enclosing volume V.

Sol^n: Here $\nabla(x^2+y^2+z^2) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)$

$$= 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$= 2(x\hat{i}+y\hat{j}+z\hat{k}).$$

$$\therefore \iint_S \nabla(x^2+y^2+z^2)\cdot d\vec{S} = \iint_S \nabla(x^2+y^2+z^2)\cdot\hat{n}\, ds$$

$$= \iint_S 2(x\hat{i}+y\hat{j}+z\hat{k})\cdot\hat{n}\, ds$$

$$= 2\iiint_V \operatorname{div}(x\hat{i}+y\hat{j}+z\hat{k})\, dv,$$

[using Divergence theorem]

Now, $\text{div}(x\hat{i}+y\hat{j}+z\hat{k}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x\hat{i}+y\hat{j}+z\hat{k})$

$$= \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}$$

$$= 3 \quad \text{——— (2)}$$

from (1) and (2), we have

$$\iint_S \nabla(x^2+y^2+z^2)\,ds = 2\iiint_V 3.\,dv$$

$$= 6\iiint_V dv$$

$$= 6V.$$

Ex:23: Use Gauss divergence theorem to evaluate $\iint_S \vec{F}.d\vec{s}$, where, $\vec{F} = 4x\hat{i}-2y^2\hat{j}+z^2\hat{k}$ and S is the surface bounding the region $x^2+y^2=4$, $z=0$ and $z=3$.

Soln: By Divergence theorem,

$$\iint_S \vec{F}.ds = \iiint_V \text{div}\,\vec{F}.dv$$

$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(4x\hat{i}-2y^2\hat{j}+z^2\hat{k})\,dv$$

$$= \iiint_V (4-4y+2z).\,dv$$

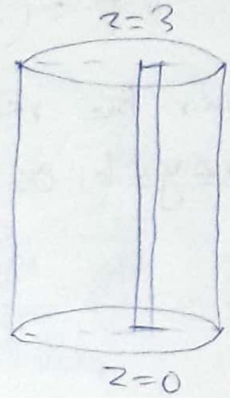
$$= \iiint (4-4y+2z)\,dx\,dy\,dz$$

$$= \iint dx\,dy \int_0^3 (4-4y+2z)\,dz$$

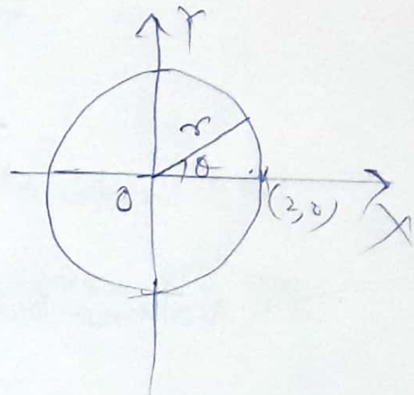
$$= \iint dx\,dy \left[4z-4yz+z^2\right]_0^3$$

$$= \iint (12-12y+9)\,dx\,dy$$

$$= \iint (21-12y)\,dx\,dy \quad \text{——— (1)}$$



$x^2+y^2=4$



Let $x = r\cos\theta$

$y = r\sin\theta$

$\therefore dx\,dy = r\,dr\,d\theta$

r varies from 0 to 2

θ varies from 0 to 2π.

$\therefore$ (1) becomes,

$$\iint_S \vec{F}\cdot ds = \int_{r=0}^{2}\int_{\theta=0}^{2\pi} (21-12r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_{r=0}^{2} (21r-12r^2\sin\theta)\cdot dr\cdot d\theta$$

$$= \int_0^{2\pi} \left[\frac{21r^2}{2} - 4r^3\sin\theta\right]_0^2 \cdot d\theta$$

$$= \int_0^{2\pi} (42-32\sin\theta)\,d\theta$$

$$= \left[42\theta + 32\cos\theta\right]_0^{2\pi}$$

$$= 84\pi - 32 + 32 = 84\pi.$$

Ex: 24 : Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

Solⁿ: We have.

$$\operatorname{div}\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left\{(x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}\right\}$$

$$= \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy)$$

$$= 2x + 2y + 2z$$

$$= 2(x+y+z).$$

$\therefore$ volume Integral $= \iiint_V \nabla\cdot\vec{F}\,dv$

$$= \iiint_V 2(x+y+z)\,dv$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c}(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c}(x+y+z)\,dz\,.\,dy\,.\,dx$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\left[xz + yz + \frac{z^2}{2}\right]_0^c .\,dy\,dx$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\left(cx + cy + \frac{c^2}{2}\right).\,dy.\,dx$$

$$= 2\int_0^a \left[cxy + c\frac{y^2}{2} + \frac{c^2y}{2}\right]_0^b dx$$

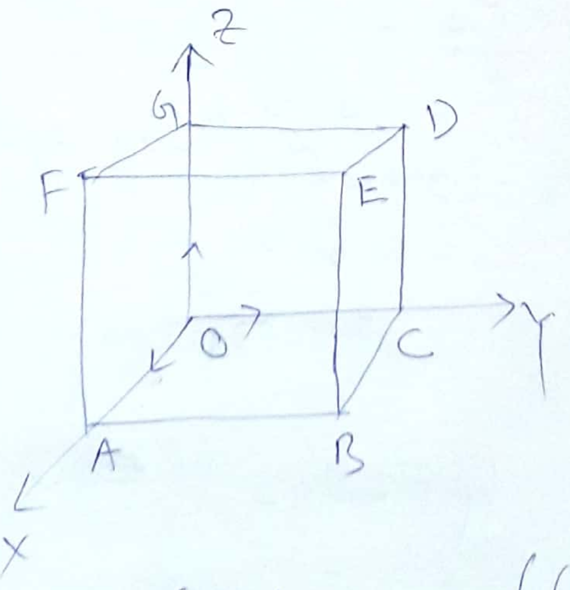
$$= 2\int_0^a \left(bcx + \frac{b^2c}{2} + \frac{bc^2}{2}\right) dx$$

$$= 2\left[bc\frac{x^2}{2} + \frac{b^2c}{2}x + \frac{bc^2}{2}x\right]_0^a$$

$$= 2\left[bc\cdot\frac{a^2}{2} + \frac{b^2c\,a}{2} + \frac{bc^2}{2}a\right]$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc(a+b+c).$$



$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{OABC} \vec{F}\cdot\hat{n}\,ds + \iint_{DEFG} \vec{F}\cdot\hat{n}\,ds$$

$$+ \iint_{OAFG} \vec{F}\cdot\hat{n}\,ds + \iint_{BCDE} \vec{F}\cdot\hat{n}\,ds + \iint_{ABEF} \vec{F}\cdot\hat{n}\,ds$$

$$+ \iint_{OCDG} \vec{F}\cdot\hat{n}\,ds$$

Surface	outward normal	ds	
OABC	$-\hat{k}$	dxdy	$z=0$
DEFG	$\hat{k}$	dxdy	$z=c$
OAFG	$-\hat{j}$	dxdz	$y=0$
BCDE	$\hat{j}$	dxdz	$y=b$
ABEF	$\hat{i}$	dydz	$x=a$
OCDG	$-\hat{i}$	dydz	$x=0$.

On OABC,

$$\iint_{OABC} \vec{F}\cdot\hat{n}\,ds = \iint_{OABC}\left\{(x^2-yz)\hat{i}+(y^2-xz)\hat{j}+(z^2-xy)\hat{k}\right\}(-\hat{k})\cdot dx\,dy$$

$$= -\int_0^a\int_0^b (z^2-xy)\cdot dx\,dy$$

$$= -\int_0^a\int_0^b (-xy)\,dx\,dy \quad [\because z=0]$$

$$= \frac{a^2b^2}{4}. \qquad \text{——— (1)}$$

On DEFG,

$$\iint_{DEFG} \vec{F}\cdot\hat{n}\,ds = \iint_{DEFG}\left\{(x^2-yz)\hat{i}+(y^2-xz)\hat{j}+(z^2-xy)\hat{k}\right\}\hat{k}\,dx\,dy$$

$$= \int_0^a\int_0^b (z^2-xy)\,dx\,dy$$

$$= \int_0^a\int_0^b (c^2-xy)\,dx\,dy$$

$$= \int_0^a\left[c^2y-\frac{xy^2}{2}\right]_0^b\cdot dx$$

$$= \int_0^a\left(bc^2-\frac{xb^2}{2}\right)\cdot dx = \left[bc^2x-\frac{x^2b^2}{4}\right]_0^a$$

$$= abc^2-\frac{a^2b^2}{4}$$

iii. $\iint_{OAFG} \vec{F}\cdot\hat{n}\,ds = \iint_{OAFG} \{(x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}\}(-\hat{j})\,dx\,dz$

$$= -\iint_{OAFG} (y^2-zx)\,dx\,dz$$

$$= -\int_0^a\int_0^c (0-zx)\,dx\,dz$$

$$= -\int_0^a \left[\frac{xz^2}{2}\right]_0^c dx$$

$$= -\int_0^a \frac{xc^2}{2}\,dx$$

$$= -\left[\frac{c^2x^2}{4}\right]_0^a = \frac{a^2c^2}{4} \quad \text{——— (3)}$$

$\iint_{BCDE} \vec{F}\cdot\hat{n}\,ds = \iint_{BCDE} \{(x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}\}\hat{j}\,dx\,dz$

$$= \iint_{BCDE} (y^2-xz)\,dx\,dz$$

$$= \int_0^a\int_0^c (b^2-xz)\,dx\,dz$$

$$= \int_0^a \left[b^2z - x\frac{z^2}{2}\right]_0^c dx$$

$$= \int_0^a \left(b^2c - \frac{xc^2}{2}\right)dx$$

$$= \left[b^2cx - \frac{x^2c^2}{4}\right]_0^a$$

$$= ab^2c - \frac{a^2c^2}{4} \quad \text{——— (4)}$$

$$\iint_{ABEF} \vec{F}\cdot\hat{n}\,ds = \iint_{ABEF} \left\{(x^2-yz)\hat{i} + (y^2-xz)\hat{j} + (z^2-xy)\hat{k}\right\}\cdot\hat{i}\,dy\,dz$$

$$= \iint_{ABEF} (x^2-yz)\,dy\,dz$$

$$= \int_{y=0}^{b}\int_{z=0}^{c} (a^2-yz)\,dy\,dz \quad [\because x=a]$$

$$= \int_0^b \left[a^2z - \frac{yz^2}{2}\right]_0^c dy$$

$$= \int_0^b \left(a^2c - \frac{yc^2}{2}\right)dy$$

$$= \left[a^2cy - \frac{c^2y^2}{4}\right]_0^b$$

$$= a^2bc - \frac{c^2b^2}{4} \quad \text{---} (5)$$

$$\iint_{OCDG} \vec{F}\cdot\hat{n}\,ds = \iint_{OCDG} \left\{(x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}\right\}(-\hat{i})\,dy\,dz$$

$$= \int_{y=0}^{b}\int_{z=0}^{c} (x^2-yz)\cdot dy\,dz$$

$$= \int_{y=0}^{b}\int_{z=0}^{c} (-yz)\,dz\cdot dy$$

$$= -\int_0^b \left[\frac{yz^2}{2}\right]_0^c\cdot dy$$

$$= -\int_0^b \frac{yc^2}{2}\cdot dy = \frac{b^2c^2}{4} \quad \text{---} (6)$$

$$\therefore \iint \vec{F}\cdot\hat{n}\,ds = \left(\frac{a^2b^2}{4}\right) + \left(abc^2 - \frac{a^2b^2}{4}\right) + \left(\frac{a^2c^2}{4}\right) + \left(ab^2c - \frac{a^2c^2}{4}\right) + \left(\frac{b^2c^2}{4}\right) + \left(a^2bc - \frac{b^2c^2}{4}\right)$$

$$= abc^2 + ab^2c + a^2bc = abc(a+b+c).$$

$\therefore$ Gauss theorem is verified.